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NONSTATIONARY PROBLEMS OF THE QUASISTATIC THEORY OF A HARDENING ELASTIC BODY

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The term nonstationary problem in the work is used in the sense proposed in [1]. This pertains to the study of slow evolutionary processes of the deformation of a rigid–plastic medium. Some experience in solving such problems has been accumulated in the theory of plastic working of porous and powdered materials. The approaches developed in the theory of compressible plastic media make it possible to take a new look at the solution of classical problems of the theory of plasticity, considering an incompressible material as a material of an extremely low compressibility. The variational principles of the theory of plastic flow are used in this work to find the velocity fields.

1. Distinctive Features of the Evolutionary Problems of the Theory of Plasticity. The simplest scheme for solving problems in which the evolution of the plastic deformation of a material during loading must be traced was presented, e.g., in [1]. If the flow pattern is to be explained, the load parameter must be changed by a step Δt, the instantaneous velocities must be found in the bulk of the material, changes in the flow geometry must be and hardening parameters must be traced, the velocities must be found again, and so forth. The main obstacles to the implementation of this approach stem from the fact that the velocity field in a rigid–plastic material is determined, generally speaking, in a nonunique manner. In numerical simulation this nonuniqueness manifests itself as incorrectness, i.e., small changes in the problem parameters result in large changes in the velocity field. For example, in the problem of the penetration of a die into a rigid–plastic homogeneous incompressible half-space an arbitrarily small variation of the yield point in the region adjacent to the die changes the velocity of the material abruptly. The velocity field symmetric relative to the die can, e.g., go over into an asymmetric field. A fair number of such examples was given in [2]. The incorrectness, in turn, entails indeterminacy in the choice of the step Δt. Of course, the faster the process, the smaller the step necessary for tracking its evolution. If the velocities change instantaneously, Δt should be zero. We arrive at this conclusion if we consider a rigid–plastic material as the limiting case of viscoplastic material. We give the rheological relations for a viscoplastic material in the form

$$e_{ij} = \beta\langle\Phi\left(\frac{F}{F_0}\right)\rangle\frac{\partial Q}{\partial\sigma_{ij}},$$

where Q is the viscoplastic potential; F is a function specifying the load surface; F_0 is a normalization factor; and the symbol $\langle\rangle$ denotes a step function of the form

$$\langle u\rangle = \begin{cases} 0 & \text{for } u < 0, \\ u & \text{for } u \geqslant 0. \end{cases}$$

For a viscoplastic material the time step should not exceed Δt_{max} [4],

$$\Delta t < \Delta t_{\max} = \frac{4(1+\nu)Y}{3\beta E} \tag{1.1}$$

(E and ν are Young's modulus and Poisson's ratio of the material in the elastic state). An estimate was made a viscoplastic potential Q = F,

$$Q = \sqrt{\tau^2} - Y,$$
$$\Phi(F/F_0) = F/Y,$$

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which corresponds to the Mises condition. For a plastic material the stresses are on the load surface (F = 0) and, therefore, for zero strain rates ($\gamma = \infty$ and $\Delta t_{max} = 0$).

2. Influence of Inclusion of Irreversible Compressibility on the Distinctive Features of the Plastic Flow of a Material. A number of papers [5-8] that have appeared despite the difficulties use the above approach to simulate considerable plastic strains of a material. The numerical results obtained in many technological problems agree qualitatively, and sometimes quantitatively, with experimental data.

The above papers studied the irreversible deformation of compressible plastic media. The advantages gained by taking compressibility into account can easily be understood on the example of the problem of the plastic deformation of a panel. Finding the velocity field in this problem reduced to minimization of the functional [9]

$$D = \int_0^1 \int_{-1}^1 \left[(u'_x)^2 + \frac{1}{\varepsilon^4}(v'_y)^2 + \frac{1}{2\varepsilon^2}(u'_y + v'_x)^2\right]^{1/2} dxdy - G(u,v),$$

where L is the length of the panel; 2h is the thickness; x and y are dimensionless coordinates; u and v are the dimensionless components of the velocity vector; G(u, v) is a linear functional in u and v (dimensionless power of the given forces acting on the panel); and the small parameter $\varepsilon = h/L$.

To within a factor the functional is an expression of the dissipative function for a material with the Mises plasticity condition and can be written as

$$D = \int_0^1 \int_{-1}^1 \sqrt{\gamma^2} dxdy - G(u,v)$$

(γ is the second invariant of the deviator of the strain-rate tensor). In the velocity field satisfying the Kirchhoff–Love conditions and the incompressibility of the medium [9]

$$u = P_0 u_0(x) + P_1(y) u_1(x),$$

$$v = -\sqrt{3} P_0 \int_0^x u_1(\lambda) d\lambda - \varepsilon^2 u'_0(x) P_0 (y + 1) - \varepsilon^2 u'_1(x) \int_{-1}^y P_1(\lambda) d\lambda$$

the functional becomes

$$\tilde{D} = \int_0^1 \int_{-1}^1 \{\sqrt{2}\,|P_0 u'_0(x) + P_1 u'_1(x)| - (q_1^0 - q_2^0)(P_0 u'_0(x) + P_1 u'_1(x))\} dxdy$$

(P_0 and P_1 are the first two polynomials orthonormalized on the segment [−1, 1] of the system of Legendre polynomials). For any finite-dimensional approximation of the velocities the minimization of such a functional reduces to solving a set of linear programming problems which are characterized by an incorrectness at certain values of the parameters. The incorrectness in the problems of finding the velocity field for plates of equal strength was noted in [10].

For a compressible material the velocity field can be written as [11]

$$u = P_0 u_0(x) + P_1(y) u_1(x), \quad v = -\sqrt{3} P_0 \int_0^x u_1(\lambda) d\lambda + \varepsilon^2 \omega(x,y).$$

The dissipative potential in the model of a compressible plastic medium, proposed by Shtern [12], has the form

$$D_c = \int_0^1 \int_{-1}^1 Y\sqrt{1-\theta}\,(\varphi\gamma^2 + \psi e^2)^{1/2} dxdy - G(u,v), \tag{2.1}$$

where Y is a constant in the Mises flow condition for a nonporous material; and φ and ψ are functions of the porosity

$$\varphi = (1 - \theta)^2, \quad \psi = \frac{2}{3}\frac{(1-\theta)^3}{\theta}.$$

Further, if Y does not depend on the space coordinates, we assume these constants to be equal to unity. On substituting the expressions for the velocity, we obtain

$$D_c = \int_0^1 \int_{-1}^1 \sqrt{1-\theta}\left[\varphi\left(u'_0(x)P_0 + u'_1(x)P_1\right)^2 + \varphi\left(\frac{\partial\omega}{\partial y}\right)^2 + \psi\left(u'_0(x)P_0 + u'_1(x)P_1 + \frac{\partial\omega}{\partial y}\right)^2\right]^{1/2} dxdy - G.$$

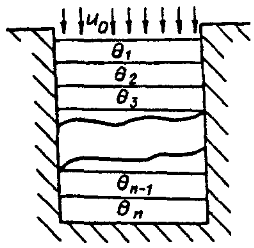

Fig. 1

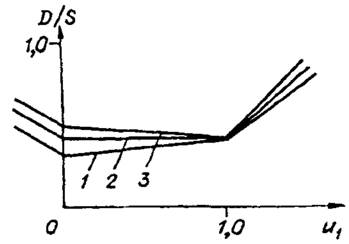

Fig. 2

We can easily ascertain that for no values of θ is the radicand a complete square and the functional does not have linear segments, i.e., the velocity field is always unique. The conclusion from the example considered above can be formulated as follows: Elimination of the linear condition of incompressibility and inclusion of the irreversible volume strain improves the properties of the functional and in many cases make it possible to obtain the correct formulation of the problem.

3. Regularization of the Problem of Finding the Velocity Field for Transient Processes. Unfortunately, an improvement such as described above, does not always occur since a functional of the form (2.1) (a specific range of integration in each specific case) is not strictly convex. Forms of deformation for which the velocity fields of the compressed plastic medium are not unique are also possible. The simplest example is the problem of unidirectional compression of a multilayer workpiece in a cylindrical die without friction (Fig. 1). The velocities at the upper and lower ends of the workpieces u_0 and zero, respectively. If we assume that all the layers are made of the same material and differ only as to porosity, the velocity distribution inside the material can be found by minimizing the functional

$$D = \sum_{i=1}^{N} \sqrt{1-\theta_i}\sqrt{\psi_i e_i^2 + \varphi_i \gamma_i^2}\, V_i = \sum_{i=1}^{N} \sqrt{1-\theta_i}\, |e_i| \sqrt{\psi_i + \frac{2}{3}\varphi_i}\, V_i.$$

Here N is the number of layers; V_i is the volume of the i-th layer; e_i and γ_i are, respectively, the first invariant of the strain-rate tensor and the second invariant of the deviator of that tensor. By u_i we denote the dimensionless velocity of the i-th layer interface ($u_0 = -1$). The functional in terms of the velocity is written as

$$D = S\sum_{i=1}^{N+1} |u_{i+1} - u_i| \sqrt{(1-\theta_i)\left(\psi_i + \frac{2}{3}\varphi_i\right)}$$

(S is the area of the die cross section). The exact solution of the problem is found simply and the gist of its sense is that only the layer with greatest porosity deforms. If the porosity of these layers is different, then

$$u_i = \prod_{j=1}^{i+1} \operatorname{sign}(\theta_i - \max_{i=1,N} \theta_i).$$

No unique velocity field in which a minimum is reached exists if the porosity of any two layers is the same. Figure 2 shows the $D(u_1)$ for a two-layer workpiece for $\theta_1 = 0.55$, $\theta_2 = 0.5$ (curve 1), $\theta_1 = 0.5$, $\theta_2 = 0.5$ (curve 2), and $\theta_1 = 0.45$, $\theta_2 = 0.5$ (curve 3). Small changes in the porosity from the state $\theta_1 = \theta_2$ result in an abrupt change in velocity. In practical numerical simulation all u_i take on values of 0 or -1, since the equal-porosity condition is never realized exactly. In other words, if we try to model the compaction of a homogeneous rod in a die by breaking it up layer by layer into elements, then the calculations are more accurate when we move away further from the actual velocity distribution, i.e., from uniform deformation of the entire rod. Henceforth we use MW to refer to the problem of compaction of a multilayer workpiece with layers of similar density. Remaining within the framework of the rigid–plastic model, we cannot avoid situations similar to those described above. The only way out is to regularize the problem by finding anew velocity field.

The most common method of regularization is that of introducing physical viscosity [3, 8]. The time step and the coefficient of viscosity are related by an equation of the type of (1.1). If the properties of the material are close to those of a rigid–plastic material, the time step should be small; this sometimes does not accord with the physical essence of the problem, e.g., in the simulation of compression of a porous rod. In another approach, which is proposed below, numerical algorithms containing regularization parameters explicitly or implicitly are used to find the velocity field.

Essentially just such an algorithm was used in [5-8]. The velocity field was determined by a method which we call the method of viscous approximations. Its content is expounded most conveniently within the framework of the variational approach. Not restricting the discussion to a rigid–plastic material, we consider a more general case for which the functional, reaching a minimum for the actual velocity field [13], has the form

$$D = \frac{1}{2\alpha}\int_{\Omega} (1 - \theta)\sigma_{\text{T}} W^{\alpha} d\Omega - G(u,v,\omega), \quad W = (\psi e^2 + \varphi\gamma^2)/(1 - \theta) \tag{3.1}$$

(ψ, φ, σ_{T} are given functions of the hardening parameters, $0.5 \le \alpha \le 1$). For a rigid–plastic material $\alpha = 0.5$ and if $\psi = \infty$ ($\theta = 0$), we obtain a Mises material. We define D_V as

$$D_V = \frac{1}{2\alpha}\int_{\Omega} \sigma_{\text{T}}(1 - \theta)\left[\alpha\frac{W}{W_0^{1-\alpha}} + (1 - \alpha)W_0^{\alpha}\right] d\Omega - G(u,v,\omega)$$

(W_0 is the value of W for given e_0 and γ_0). For all e, γ we have

$$D \leqslant D_V. \tag{3.2}$$

The values of the functionals D and D_V coincides only for $e = e_0$, $\gamma = \gamma_0$. The proof follows from the inequality

$$\left(\frac{W}{W_0}\right)^{\alpha} \leqslant 1 + \alpha\frac{W}{W_0},$$

which was obtained by comparing the integrands in D and D_V. The validity of the inequality is obvious because of the convexity of the step function in the indicated range of α. From Eq. (3.2) we have

$$D(e_1,\gamma_1) \leqslant \min D_V \leqslant D(e_0,\gamma_0),$$

where $D(e_1, \gamma_1)$ is the value of the functional, calculated from the velocities, that minimizes the quadratic functional D_V. The equal sign appears only at the minimum. Substituting the new values e_1, γ_1 instead of e_0, γ_0 and repeating the procedure the required number of times, we obtain an iteration process, which always converges to the desired new velocity field. A similar proof of the viscous approximations method can be constructed for any convex functional, in which case an inequality analogous to (3.2) follows from the properties of the subdifferential [13]. The functional D_V is called the viscous approximation of the functional D.

The condition of practical convergence of the velocities obtained is used in [5-8] as a test that a minimum is reached. If the properties of the material are close to rigid-plastic (the functional is not strictly convex), it can be proved that an arbitrary small constant in the practical convergence condition of the viscous approximations method does not guarantee that the position of the functional minimum can be determined with sufficient accuracy. At the same time, as was shown above in the MW problem, an exact solution may be uninformative because of the shortcomings of the model itself. The constant in the practical convergence condition, therefore, serves rather as a regularization parameter, which indicates how close the values of the dissipative functionals of the rigid–plastic material considered should be at the end of the iteration process to those of the equivalent viscous material, obtained by means of successive viscous approximations. A shortcoming of such regularization is that the velocities obtained have no comprehensible physical meaning since the iteration process with various initial approximations can give sharply differing results.

A more promising approach is that of constructing numerical algorithms in which information about the velocity field would be used not only at some given time t_0 but also over an entire interval $[t_0, t_0 + \Delta t]$, at the time $t_0 + \Delta t$ in the simplest case. With more complete information about the flow of the material the average velocities at time t_0. In a numerical analysis of processes modeled by differential equations, the use of implicit schemes makes averaging possible. The model of rigid–plastic material contains no information about the evolutionary equations for the velocities. For deferential functionals they can be obtained by differentiating the pertinent Euler equations with respect to time. In our case this is impossible and, moreover, the very idea of an implicit approach requires amendment. For example, we organize the following iteration process: from a given initial approximation of the velocity field and the initial values of the hardening parameters we determine the parameters of the material at time $t_0 + \Delta t$, find the value, and return to the beginning of the process. As the average velocity field over the interval under consideration we take the velocity field to which the iteration process converges. This implicit approach turns out to have the disadvantage that velocities giving the exact solution of the problem of deformation of the

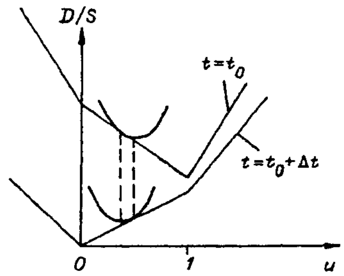

Fig. 3

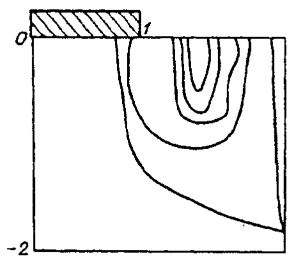

Fig. 4

rigid−plastic material must be used. This is why the iteration process diverges, e.g., in the MW problem for which the sought velocity cannot be obtained by numerical analysis of the rigid−plastic model.

It proved possible to construct an effective numerical method on the basis of a combination of the viscous approximations method and the use of implicit schemes. Its essence consists in the fact that an iteration process similar to that above is used for each viscous approximation and not for the limiting rigid−plastic material. We differentiate the functional D_v; for velocity fields minimizing this functional, we can obtain evolutionary equations, using ordinary numerical treatments to reduce them to a system of ordinary differential equations and then to solve those equations by means of the implicit schemes. In the simplest case, as already indicated, we can confine the discussion to the iteration procedure described above, successively going from one viscous approximation to another. If in the material Δt the velocity varied stably, the velocity obtained by the implicit method differs that $u(t_0)$ and $u(t_0 + \Delta t)$ by a value of the order of $O(\Delta t)$. If the velocity varied unstably and changed by a finite value in the time Δt, the iteration process converges to the average velocity. This velocity field minimizes the quadratic functional, which on the one hand approximates the initial functional at the time $t_0 + \Delta t$ and, on the other hand, cannot be improved at the time t_0 without worsening the approximation at the time $t_0 + \Delta t$. More exactly, for the velocity field found the change in velocity $\Delta \mathbf{u}_0$ in a step of the viscous approximation method at t_0 is compensated with a given accuracy by a change in velocity $\Delta \mathbf{u}_1$ in a similar step of the viscous approximations method but at the time $t_0 + \Delta t$. The condition for stopping the iteration is

$$|\Delta u_0 + \Delta u_1| < \varepsilon.$$

If the instantaneous velocities changed abruptly in the interval of time considered, then $\Delta \mathbf{u}_0$ and $\Delta \mathbf{u}_1$ remain finite values even for arbitrarily small Δt. For the compaction of a two-layer workpiece the initial and approximating quadratic functional are represented schematically in Fig. 3. The time when the iteration process stops is shown. In the approach studied Δt is a regularization parameter and is chosen on the basis of physical ideas concerning the characteristic rates of change of the parameters. The rate at which the iterations converge is determined by the rates of convergence of the viscous approximations method at the times t_0 and $t_0 + \Delta t$. If at t_0 the method of viscous approximations converges slowly, the entire iteration process can converge rapidly because of the better convergence at $t_0 = \Delta t$, and conversely. Generally speaking, the convergence of the iteration process is better when the time step is larger and D is more sensitive to the hardening parameters.

As an example, we solve the problem of the penetration of a die into a rigid−plastic finite strip with a width equal to that of the die (plane deformation). All the contact surfaces are assumed to be absolutely smooth. The material of the strip is weakly compressed, with a uniformly distributed porosity of 0.001. The hardening law in Eq. (3.1) is given in the form usually taken for iron,

$$\sigma_T = 140 + 50\,\Gamma_0^{1/2},\ \text{MPa};\ \frac{d\Gamma_0}{dt} = \sqrt{W}$$

(Γ_0 is an analog of the Odqvist parameter for a compressed plastic solid [12]). At the initial time $\Gamma_0 = 0$ in the entire volume. The constant in the practical convergence condition is $\varepsilon = 0.001$. The region considered was divided into 126 finite elements, at whose sites 137 velocities were determined from the mimimum condition (3.1). The total number of iterations per step is 30. Figure 4 shows isolines of the field of the quantity

$$\xi = \frac{p}{2\sigma_T} - \alpha,\ \alpha = 0.5\,\mathrm{arctg}\,\frac{e_y - e_x}{2e_{xy}}, \tag{3.3}$$

where p is the pressure in the material, which was found from the formula [12] (this quantity was not determined for an incompressible material)

$$p = \sigma_т \sqrt{1 - \theta} \frac{(\psi + \varphi/6)e/\gamma}{\sqrt{\varphi + \psi e^2/\gamma^2}}, \; p = \frac{\sigma_x + \sigma_y}{2}.$$

We see that the isolines differ substantially from the family of slip lines for an ideal plastic material, the equation for which is (3.3). In particular, a large region of uniformly deformed material is formed beneath the die. This solution cannot be obtained if the velocity field is considered only at the initial time. In this case the functional should reach a minimum at the breaking velocity fields (Prandtl and Hill solutions [14]), which is impossible for a hardenable material [15].

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